

# Semiparametric Single Index Models

Li and Racine (2007, Chapter 8)

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December 16, 2024

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- A semiparametric single index model is given by

$$Y = g(X^T \beta_0) + u,$$

where

$Y \in \mathbb{R}$  : a dependent variable,

$X \in \mathbb{R}^q$  : a  $q \times 1$  explanatory vector,

$\beta_0 \in \mathbb{R}^q$  : a  $q \times 1$  vector of unknown parameters,

$u \in \mathbb{R}$  : an error term which satisfies  $\mathbb{E}(u \mid X) = 0$ ,

$g(\cdot)$  : an unknown distribution function.

- Even though  $x$  is a  $q \times 1$  vector,  $x^T \beta_0$  is a scalar of a single linear combination, which is called **a single index**.
- By the form of the single index model, we obtain

$$\mathbb{E}(Y \mid X) = g(X^T \beta_0),$$

which means that the conditional expectation of  $Y$  only depends on the vector  $X$  through a single index  $X^T \beta_0$ .

- The model is semiparametric when  $\beta \in \mathbb{R}^q$  is estimated with the parametric methods and  $g(\cdot)$  with the nonparametric methods.

# Examples of Parametric Single Index Model

- If  $g(\cdot)$  is the identity function, then the model turns out to be a linear regression model:

$$Y = g(X^T \beta_0) + u = X^T \beta_0 + u.$$

- If  $g(\cdot)$  is the CDF of Normal(0, 1), then the model turns out to be a probit model.
  - See the textbook for further discussions on a probit model.
- If  $g(\cdot)$  is the CDF of logistic distribution, then the model turns out to be a logistic regression model.

Identification Conditions

Ichimura's (1993) Method

Direct Semiparametric Estimators for  $\beta$

Bandwidth Selection

Klein and Spady's (1993) Estimator

Lewbel's (2000) Estimator

Manski's (1975, 1985) Maximum Score Estimator

Horowitz's (1992) Smoothed Maximum Score Estimator

Han's (1987) Maximum Rank Estimator

Multinomial Discrete Choice Models

Ai's (1997) Semiparametric Maximum Likelihood Approach

References

# Identification Conditions

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# Identification Conditions

## Proposition 8.1 (Identification of a Single Index Model)

For the semiparametric single index model  $Y = g(x^T \beta_0) + u$ , identification of  $\beta_0$  and  $g(\cdot)$  requires that

- (i)  $x$  should not contain a constant/an intercept, and must contain at least one continuous variable. Moreover,  $\|\beta_0\|=1$ .
- (ii)  $g(\cdot)$  is differentiable and is not a constant function on the support of  $x^T \beta_0$ .
- (iii) For the discrete components of  $x$ , varying the values of the discrete variables will not divide the support of  $x^T \beta_0$  into disjoint subsets.

## Identification Condition (i)

- Note that the location and the scale of  $\beta_0$  are not identified.
- The vector  $x$  cannot include an intercept because the function  $g(\cdot)$  (which is to be estimated in nonparametric manners) includes any location and level shift.
  - That is,  $\beta_0$  cannot contain a location parameter.

## Identification Condition (i)

- Some normalization criterion (scale restrictions) for  $\beta_0$  are needed.
  - One approach is to set  $\|\beta_0\| = 1$ .
  - The second approach is to set one component of  $\beta_0$  to equal one. This approach requires that the variable corresponding to the component set to equal one to be continuously distributed and has a non-zero coefficient.
  - Then,  $x$  must be dimension 2 or larger. If  $x$  is one-dimensional, then  $\beta_0 \in \mathbb{R}^1$  is simply normalized to 1, and the model is the one-dimensional nonparametric regression  $E(Y | x) = g(x)$  with no semiparametric component.

## Identification Conditions (ii) and (iii)

- The function  $g(\cdot)$  cannot be a constant function and must be differentiable on the support of  $x^T \beta_0$ .
- $x$  must contain at least one continuously distributed variable and this continuous variable must have non-zero coefficient.
  - If not,  $x^T \beta_0$  only takes a discrete set of values and it would be impossible to identify a continuous function  $g(\cdot)$  on this discrete support.

## Ichimura's (1993) Method

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# Ichimura's Method

- Textbook: Sections 8.2; 8.4.1; and 8.12.
- Suppose that the functional form of  $g(\cdot)$  were known.
- Then we could estimate  $\beta_0$  by minimizing the least-squares criterion:

$$\sum_{i=1} [Y_i - g(X_i^T \beta)]^2$$

with respect to  $\beta$ .

- We could think about replacing  $g(\cdot)$  with a nonparametric estimator  $\hat{g}(\cdot)$ .
- However, since  $g(z)$  is the conditional mean of  $Y_i$  given  $X_i^T \beta_0 = z$ ,  $g(\cdot)$  depends on unknown  $\beta_0$ . thus we cannot estimate  $g(\cdot)$  here.

- Nevertheless, for a fixed value of  $\beta$ , we can estimate

$$G(X_i^T \beta) := \mathbb{E}(Y_i \mid X_i^T \beta) = \mathbb{E}(g(X_i^T \beta_0) \mid X_i^T \beta).$$

- In general  $G(X_i^T \beta) \neq g(X_i^T \beta)$ .
- When  $\beta = \beta_0$ , it holds that  $G(X_i^T \beta_0) = g(X_i^T \beta_0)$ .<sup>1</sup>

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<sup>1</sup>一般の  $X_i^T \beta$  を用いて条件付けると、 $G$  と  $g$  は通常は一致しないが、正しいインデックス  $X_i^T \beta = X_i^T \beta_0$  のときだけ一致するということ。

- First, we estimate  $G(X_i^T \beta)$  with the leave-one-out NW estimator:

$$\begin{aligned}\hat{G}_{-i}(X_i^T \beta) &:= \hat{\mathbb{E}}_{-i}(Y_i \mid X_i^T \beta) \\ &= \frac{\sum_{j \neq i} Y_j K\left(\frac{X_j^T \beta - X_i^T \beta}{h}\right)}{\sum_{j \neq i} K\left(\frac{X_j^T \beta - X_i^T \beta}{h}\right)}.\end{aligned}$$

# Ichimura's Method

- Second, using the leave-one-out NW estimator  $\hat{G}_{-i}(X_i^T \beta)$ , we estimate  $\beta$  with

$$\begin{aligned}\hat{\beta} &:= \arg \min_{\beta} \sum_{i=1}^n \left[ Y_i - \hat{G}_{-i}(X_i^T \beta) \right]^2 w(X_i) \mathbf{1}(X_i \in A_n) \\ &:= \arg \min_{\beta} S_n(\beta),\end{aligned}$$

which is called **Ichimura's estimator (the WSLs estimator)**.

- $w(X_i)$  is a nonnegative weight function.
- $\mathbf{1}(X_i \in A_n)$  is a trimming function to trim out small values of  $\hat{p}(X_i^T \beta) = \frac{1}{nh} \sum_{j \neq i} K \left( \frac{X_j^T \beta - X_i^T \beta}{h} \right)$ , so that we do not suffer from the random denominator problem.
  - $A_\delta = \{x : p(x^T \beta) \geq \delta, \text{ for } \forall \beta \in \mathcal{B}\}$ .
  - $A_n = \{x : \|x - x^*\| \leq 2h, \text{ for } \exists x^* \in A_\delta\}$ , which shrinks to  $A_\delta$  as  $n \rightarrow \infty$  and  $h \rightarrow 0$ .

# Asymptotic Distribution of Ichimura's Estimator

- Let  $\hat{\beta}$  denote the semiparametric estimator of  $\beta_0$  obtained from minimizing  $S_n(\beta)$ .
- To derive the asymptotic distribution of  $\hat{\beta}$ , the following conditions are needed:

# Asymptotic Distribution of Ichimura's Estimator

## Assumption 8.1

The set  $A_\delta$  is compact, and the weight function  $w(\cdot)$  is bounded and positive on  $A_\delta$ . Define the set

$$D_z = \{z : z = x^T \beta, \beta \in \mathcal{B}, x \in A_\delta\}.$$

Letting  $p(\cdot)$  denote the PDF of  $z \in D_z$ ,  $p(\cdot)$  is bounded below by a positive constant for  $\forall z \in D_z$

## Assumption 8.2

$g(\cdot)$  and  $p(\cdot)$  are 3 times differentiable w.r.t.  $z = x^\beta$ . The third derivatives are Lipschitz continuous uniformly over  $\mathcal{B}$  for  $\forall z \in D_z$ .

# Asymptotic Distribution of Ichimura's Estimator

## Assumption 8.3

The kernel function is a bounded second order kernel, which has bounded support; is twice differentiable; and its second derivative is Lipschitz continuous.

## Assumption 8.4

$\mathbb{E}(|Y^m|) < \infty$  for  $\exists m \geq 3$ .  $\text{var}(Y | x)$  is bounded and bounded away from zero for  $\forall x \in A_\delta$ .  $\frac{q \ln(h)}{nh^{3+\frac{3}{m-1}}} \rightarrow 0$  and  $nh^8 \rightarrow 0$  as  $n \rightarrow \infty$ .

# Asymptotic Distribution of Ichimura's Estimator

**Theorem 8.1.** Under assumptions 8.1 through 8.4,

$$\sqrt{n}(\hat{\beta} - \beta_0) \xrightarrow{d} \text{Normal}(0, \Omega_I),$$

with

$$\Omega_I = V^{-1} \Sigma V^{-1},$$

$$V = \mathbb{E}\{w(X_i)(g_i^{(1)})^2 \\ \times (X_i - E_A(X_i | X_i^T \beta_0))(X_i - E_A(X_i | X_i^T \beta_0))^T\},$$

$$\Sigma = \mathbb{E}\{w(X_i)\sigma^2(X_i)(g_i^{(1)})^2 \\ \times (X_i - E_A(X_i | X_i^T \beta_0))(X_i - E_A(X_i | X_i^T \beta_0))^T\},$$

where

- $(g_i^{(1)}) = \frac{\partial g(v)}{\partial v} \big|_{v=X_i^T \beta_0},$
- $\mathbb{E}_A(X_i | v) = \mathbb{E}(X_i | x_A^T \beta_0 = v),$
- $x_A$  has the distribution of  $X_i$  conditional on  $X_i \in A_\delta.$

- See Ichimura (1993); and Hardle, Hall and Ichimura (1993) for the proof of **Theorem 8.1**.
- Horowitz (2009) provides an excellent heuristic outline for proving **Theorem 8.1**, using only the familiar Taylor series methods, the standard LLN, and the Lindeberg-Levy CLT.

# Optimal Weight under the Homoscedasticity Assumption

- We introduce the following homoscedasticity assumption:

$$\mathbb{E}(u_i^2 \mid X_i) = \sigma^2.$$

- Under this assumption, the optimal choice of  $w(\cdot)$  is  $w(X_i) = 1$ .
- In this case,

$$\hat{\beta} = \arg \min_{\beta} \sum_{i=1}^n (Y_i - \hat{G}_{-i}(X_i^T \beta)^2) \mathbf{1}(X_i \in A_n)$$

is **semiparametrically efficient** in the sense that  $\Omega_I$  is **the semiparametric variance lower bound** (conditional on  $X \in A_\delta$ ).

# Optimal Weight under Heteroscedasticity

- In general,  $\mathbb{E}(u_i^2 \mid X_i) = \sigma^2(X_i)$ .
- **An infeasible case:** If one assumes that  $\mathbb{E}(u_i^2 \mid X_i) = \sigma^2(X_i^T \beta_0)$ , that is, the conditional variance depends only on the single index  $X_i^T \beta_0$ , the choice of  $w(X_i) = \frac{1}{\sigma^2(X_i^T \beta_0)}$  can lead to a semiparametrically efficient estimation.
- We could employ a two-step procedure as follows.

## Two-Step Procedure to Choose Optimal Weight

- Suppose that the conditional variance is a function of  $X_i^T \beta_0$  (Let  $\sigma^2(X_i^T \beta_0)$  denote it).
- **The first step:** Use  $w(X_i) = 1$  to obtain a  $\sqrt{n}$ -consistent estimator of  $\beta_0$ .
- Let  $\tilde{\beta}_0$  denote the estimator of  $\beta_0$ , and  $\tilde{u}_i = Y_i - \hat{g}(X_i^T \tilde{\beta}_0)$  denote the residual obtained from  $\tilde{\beta}_0$ .
- We can obtain a consistent nonparametric estimator of the conditional variance:  $\hat{\sigma}^2(X_i^T \tilde{\beta}_0)$ .

## Two-Step Procedure to Choose Optimal Weight

- **The second step:** Estimate  $\beta_0$  again using  $w(X_i) = \frac{1}{\hat{\sigma}^2(X_i^T \tilde{\beta}_0)}$ :

$$\hat{\beta}_0 = \arg \min_{\beta} \sum_{i=1}^n \left[ Y_i - \hat{G}_{-i}(X_i^T \beta) \right]^2 \frac{1}{\hat{\sigma}^2(X_i^T \tilde{\beta}_0)} \mathbf{1}(X_i \in A_n).$$

- The estimator  $\hat{\beta}_0$  is semiparametrically efficient because  $\hat{\sigma}^2(v) - \sigma^2(v)$  converges to zero at a particular rate uniformly over  $v \in D_v$  ( $D_v$  is the support of  $X_i^T \beta_0$ ).<sup>2</sup>

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<sup>2</sup> $\hat{\sigma}^2(X_i^T \beta)$  を用いるケースもある.

# Direct Semiparametric Estimators for $\beta$

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## Direct Semiparametric Estimators for $\beta$

- Textbook: Sections 8.3; and 8.4.2.
- Here we review:
  - Hurdle and Stoker's (1989) Average Derivative Estimator,
  - Powell, Stock and Stoker's (1989) Density-Weighted Average Derivative Estimator,
  - Li, Lu and Ullah's (2003) Estimator, and
  - Hristache, Juditsky and Spokoiny's (2001) Improved Average Derivative Estimator.
- The advantage of the direct estimation method is that we can estimate  $\beta_0$  and  $g(x^T \beta_0)$  directly without running the nonlinear least squares, which leads to the computational simplicity.
- We still suffer from a finite-sample problem.

## Hardle and Stoker's (1989) Average Derivative Estimator

- Suppose that  $x$  is a  $q \times 1$  vector of continuous variables.
- Then we obtain **the average derivative** of  $\mathbb{E}(Y \mid X = x)$ :

$$\mathbb{E} \left[ \frac{\partial \mathbb{E}(Y \mid X = x)}{\partial x} \right] = \mathbb{E} \left[ g^{(1)}(x^T \beta_0) \right] \beta_0$$

- Recall that the scale of  $\beta_0$  is not identified, which means that the constant  $\mathbb{E} \left[ g^{(1)}(x^T \beta_0) \right]$  does not matter. That is, a normalized estimation of  $\mathbb{E} \left[ \frac{\partial \mathbb{E}(Y \mid X = x)}{\partial x} \right]$  is an estimation of normalized  $\beta_0$ .

## Hardle and Stoker's (1989) Average Derivative Estimator

- Let  $\hat{\mathbb{E}}(Y_i | X_i)$  denote the NW estimator of  $\mathbb{E}(Y_i | X_i)$ :

$$\hat{\mathbb{E}}(Y_i | X_i) = \frac{\sum_{j=1}^n Y_j K\left(\frac{X_i - X_j}{a}\right)}{\sum_{j=1}^n K\left(\frac{X_i - X_j}{a}\right)}.$$

- Assuming that the kernel function is differentiable, we can estimate  $\beta_0$ , estimating  $\mathbb{E}\left[\frac{\partial \mathbb{E}(Y|X=x)}{\partial x}\right]$  with its sample analogue:

$$\tilde{\beta}_{ave} = \frac{1}{n} \sum_{i=1}^n \frac{\partial \hat{\mathbb{E}}(Y_i | X_i)}{\partial X_i}.$$

- The scale normalization can also be implemented by  $\frac{\tilde{\beta}_{ave}}{|\tilde{\beta}_{ave}|}$ .

## Hardle and Stoker's (1989) Average Derivative Estimator

- An issue raised with this estimator is the random denominator problem, which leads to a difficulty in the derivation of the asymptotic properties.
- Rilstone (1991) establishes the  $\sqrt{n}$ -normality using a trimming function.

## Powell, Stock and Stoker's (1989) Density-Weighted Average Derivative Estimator

- As we obtain the average derivative above, we also obtain the **weighted average derivative** of  $\mathbb{E}(Y \mid X = x)$ :

$$\mathbb{E} \left[ w(x) \frac{\partial \mathbb{E}(Y \mid X = x)}{\partial x} \right] = \mathbb{E} \left[ w(x) g^{(1)}(x^T \beta_0) \right] \beta_0.$$

# Powell, Stock and Stoker's (1989) Density-Weighted Average Derivative Estimator

- Let  $w(x)$  be the density function  $f(x)$ , and  $\delta$  denote the density-weighted average derivative of  $\mathbb{E}(Y \mid X = x)$ .
- Then we obtain

$$\begin{aligned}\delta &= \mathbb{E} \left[ f(X) \frac{\partial \mathbb{E}(Y \mid X = x)}{\partial x} \right] \\ &= \mathbb{E} \left[ f(X) g^{(1)}(X^T \beta_0) \right] \\ &= \int g^{(1)}(x^T \beta_0) f^2(x) dx \\ &= g(x^T \beta_0) f^2(x) - 2 \int g(x^T \beta_0) f^{(1)}(x) f(x) dx.\end{aligned}$$

# Powell, Stock and Stoker's (1989) Density-Weighted Average Derivative Estimator

- Assume that  $f(x) = 0$  at the boundary of the support of  $X$ . Then we observe that  $g(x^T \beta_0) f^2(x) = 0$ , that is,

$$\begin{aligned}\delta &= -2 \int g(x^T \beta_0) f^{(1)}(x) f(x) dx \\ &= -2\mathbb{E}[g(X^T \beta_0) f^{(1)}(X)] \\ &= -2\mathbb{E}[Y f^{(1)}(X)].\end{aligned}$$

- We can estimate  $\delta$  by its sample analogue:

$$\hat{\delta} = -\frac{2}{n} \sum_{i=1}^n Y_i \hat{f}_{-i}^{(1)}(X_i),$$

where  $\hat{f}_{-i}(X_i)$  is the leave-one-out NW estimator of  $f(X)$ :

$$\hat{f}_{-i}(X_i) = \frac{1}{n-1} \sum_{j \neq i} \left(\frac{1}{h}\right)^q K\left(\frac{X_i - X_j}{h}\right).$$

# Powell, Stock and Stoker's (1989) Density-Weighted Average Derivative Estimator

- There is no denominator messing with uniform convergence.  
There is only a density estimator, no conditional mean needed.
- The textbook uses the NW estimator  $\hat{f}^{(1)}(X_i)$  in (8.17).
- However, Powell, Stock and Stoker (1989) define their estimator using the leave-one-out NW estimator  $\hat{f}_{-i}^{(1)}(X_i)$ .
- Here we proceed with Powell, Stock and Stoker (1989).

# Powell, Stock and Stoker's (1989) Density-Weighted Average Derivative Estimator

- A useful representation of  $\hat{\delta}$  is given by

$$\hat{\delta} = \frac{-2}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i} \left(\frac{1}{h}\right)^{q+1} Y_i K^{(1)}\left(\frac{X_i - X_j}{h}\right).$$

- Under some assumptions, if  $h \rightarrow 0$  and  $nh^{q+2} \rightarrow \infty$  hold, then the density-weighted average derivative estimator  $\hat{\delta}$  satisfies that

$$\sqrt{n}(\hat{\delta} - \mathbb{E}[\hat{\delta}]) \xrightarrow{d} \text{Normal}(0, \Sigma_{\delta}),$$

where

$$\Sigma_{\delta} = 4\mathbb{E}[\sigma^2(X)f^{(1)}(X)f^{(1)}(X)^T] + 4\text{Var}(f(X)g^{(1)}(X)).$$

## $U$ -Statistics Form of $\hat{\delta}$

- Recall that  $K(\cdot)$  is differentiable and symmetric, that is,  $K^{(1)}(u) = -K^{(1)}(-u)$ . Then, we obtain the standard  $U$ -statistics form of  $\hat{\delta}$ :

$$\hat{\delta} = - \binom{n}{2}^{-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^n \left(\frac{1}{h}\right)^{q+1} K^{(1)}\left(\frac{X_i - X_j}{h}\right) (Y_i - Y_j).$$

- Letting  $Z_i$  denote  $(Y_i, X_i^T)^T$  and  $p_n(Z_i, Z_j)$  denote  $-\frac{1}{h^{q+1}} K^{(1)}\left(\frac{X_i - X_j}{h}\right) (Y_i - Y_j)$ ,  $\hat{\delta}$  can be rewritten as

$$\hat{\delta} = \binom{n}{2}^{-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^n p_n(Z_i, Z_j).$$

- This representation of  $\hat{\delta}$  permits a direct analysis of its asymptotic properties, based on the asymptotic theory of  $U$ -statistics. Further discussions can be seen in Serfling (1980); van der Vaart (2000, Chapter 12).

# Powell, Stock and Stoker's (1989) Density-Weighted Average Derivative Estimator

- The asymptotic bias is a bit complicated.
- Let  $q$  be the dimension of  $X$ , and set

$$p = \begin{cases} \frac{q+4}{2} & \text{if } q \text{ is even,} \\ \frac{q+3}{2} & \text{if } q \text{ is odd.} \end{cases}$$

- The kernel function  $K(\cdot)$  for the estimation of  $f(\cdot)$  is required to be of order at least  $p$ .
- The asymptotic bias is  $\sqrt{n}(\mathbb{E}(\hat{\delta}) - \delta) = O(n^{\frac{1}{2}}h^p)$ , which is  $o(1)$  if  $nh^{2p} \rightarrow 0$ .

# Powell, Stock and Stoker's (1989) Density-Weighted Average Derivative Estimator

- $nh^{2p} \rightarrow 0$  is violated if  $h$  is selected to be optimal for the estimation of  $f(\cdot)$  or  $f^{(1)}(\cdot)$ . That is, this requirement needs the bandwidth  $h$  to undersmooth to reduce the bias. Further discussions on the bandwidth selection follow in Section 8.4.
- Cattaneo, Crump and Jansson (2010, 2011) introduce another asymptotic theory to relax strong assumptions .
- Nishiyama and Robinson (2005): Density-weighted average derivative estimators can be refined by bootstrapping methods.

## Abstract

In a number of semiparametric models, smoothing seems necessary in order to obtain estimates of the parametric component which are asymptotically normal and converge at parametric rate. However, smoothing can inflate the error in the normal approximation, so that refined approximations are of interest, especially in sample sizes that are not enormous. We show that a bootstrap distribution achieves a valid Edgeworth correction in case of density-weighted averaged derivative estimates of semiparametric index models. Approaches to bias-reduction are discussed. We also develop a higher order expansion, to show that the bootstrap achieves a further reduction in size distortion in case of two-sided testing. The finite sample performance of the methods is investigated by means of Monte Carlo simulations from a Tobit model.

**Keywords:** Bootstrap; Edgeworth correction; semiparametric averaged derivatives

## Li, Lu and Ullah's (2003) Estimator

- We consider the estimation of the average derivative  $\mathbb{E}[g^{(1)}(X^T \beta_0)]$  again.
- We can also use the local polynomial method for the estimation of  $g^{(1)}(X^T \beta_0)$ .
- Let  $\hat{g}^{(1)}(X_i^T \beta_0)$  denote the kernel estimator of  $g^{(1)}(X_i \beta_0)$ , which is obtained from an  $m$ -th order local polynomial regression.
- Li, Lu and Ullah (2003) suggest to use  $\tilde{\beta}_{ave} = \frac{1}{n} \hat{g}^{(1)}(X_i^T \beta_0)$  to estimate  $\beta = \mathbb{E}[g^{(1)}(X^T \beta_0)]$ .

- Their approach does not require the condition  $f(x) = 0$  at the boundary of the support of  $X$ . However, they require to assume that
  - the support of  $X$  is a compact set, and that
  - the density  $f(x)$  is bounded below by a positive constant at the support of  $X$ ,which avoids the use of a trimming function.

## Li, Lu and Ullah's (2003) Estimator

- Under the assumptions so far and some additional conditions, if we use a second order kernel, where  $n \sum_{s=1}^q a_s^{2m} \rightarrow 0$  and  $\frac{na_1 \cdots a_q \sum_{s=1}^q}{\ln(n)} \rightarrow \infty$  with  $m$  denoting the order of local polynomial estimation, then,

$$\sqrt{n}(\tilde{\beta}_{ave} - \beta) \xrightarrow{d} \text{Normal} \left( 0, \Phi + \text{var}(g^{(1)}(X^T \beta_0)) \right),$$

$$\text{where } \Phi = \mathbb{E} \left[ \frac{\sigma^2(X) f^{(1)}(X) f^{(1)}(X)^T}{f^{(2)}(X)} \right].$$

- The proof of the asymptotic normality can be derived from  $U$ -statistics theory.
- Newey (1994) shows that the asymptotic variance does not depend on the specific nonparametric estimation method.

# Hristache, Juditsky and Spokoiny's (2001) Improved Average Derivative Estimator

- Powell, Stock and Stoker's (1989) density-weighted average derivative estimator requires the density of  $X$  to be increasingly smooth as the dimension of  $X$  increases.
- This is necessary to make  $\sqrt{n}(\hat{\delta} - \delta)$  asymptotically normal with a mean of 0.
- **Practical Consequence:** The finite-sample performance of the density-weighted average derivative estimator is likely to be deteriorated as the dimension of  $X$  increases, especially if the density of  $X$  is not very smooth.
- Specifically, the estimator's bias and MSE are likely to increase as the the dimension of  $X$  increases.

# Hristache, Juditsky and Spokoiny's (2001) Improved Average Derivative Estimator

- Hristache, Juditsky and Spokoiny (2001) introduce an iterated average derivative estimator that overcomes this problem.
- Their estimator is based on the observation that  $g(x^T \beta_0)$  does not vary when  $x$  varies in a direction that is orthogonal to  $\beta_0$ .
- Therefore, only the directional derivative of  $\mathbb{E}(Y \mid X = x)$  along the direction of  $\beta$  is needed for estimation.
- Suppose that this direction were known. Then estimating the directional derivative would be a one-dimensional nonparametric estimation problem, and there would be no curse of dimensionality.

# Hristache, Juditsky and Spokoiny's (2001) Improved Average Derivative Estimator

- In practice, the direction of  $\beta$  is unknown.
- Hristache, Juditsky and Spokoiny (2001) show that this can be estimated with sufficient accuracy through an iterative procedure.
- Their idea is to use prior information about the vector  $\beta$  for improving the quality of the gradient estimate by extending a weighting kernel in the direction of small directional derivatives, and they demonstrate that the whole procedure requires at most  $2 \log(n)$  iterations.
- Under relatively mild assumptions, their estimator is  $\sqrt{n}$ -consistent.
- See Horowitz (2009, Section 2.6) for further discussions.

- アイデア自体は単純：
- $x$  が  $\beta$  と直交する方向に動いても  $x^T \beta$  の変動はゼロである．一方で， $x$  が  $\beta$  と同じ方向に動けば，推定のために用いる情報として有益である．そこで，そのような方向の  $x$  の変動のみをとってきて，1次元の話の考えるとよいのでは？
- 「捨てる」ような  $x$  の変動に対しては，バンド幅を大きくとってやる．すると，その分の収束スピードがはよくなって，「次元の呪い」はクリアできるはず．

## Estimation of $g(\cdot)$

- Let  $\beta_n$  denote a  $\sqrt{n}$ -consistent estimator of  $\beta$ , or  $\delta$ .
- Once we obtain  $\beta_n$ , we can estimate  $g(x^T \beta_0)$  by

$$\hat{g}(x^T \beta_n) = \frac{\sum_{j=1}^n Y_j K\left(\frac{(X_j - x)^T \beta_n}{h}\right)}{\sum_{j=1}^n K\left(\frac{(X_j - x)^T \beta_n}{h}\right)}.$$

- Recall that  $\beta_n$  is a  $\sqrt{n}$ -consistent estimator of  $\beta$ , that is,  $\beta_n - \beta_0 = Op(n^{-\frac{1}{2}})$ ,
- This converges to zero faster than standard nonparametric estimators.
- Then, the asymptotic distribution of  $\hat{g}(x^T \beta_n)$  is the same as that of  $\hat{g}(x^T \beta_0)$ .
- Thus, we obtain **Corollary 8.1**:

$$\sqrt{nh}[\hat{g}(x^T \beta_n) - g(x^T \beta_0) - h^2 B(x_0^\beta)] \xrightarrow{d} \text{Normal}\left(0, \frac{\kappa \sigma^2(x^T \beta_0)}{f(x^T \beta_0)}\right).$$

## Estimation of $g(\cdot)$ (補足)

- $\hat{g}(x^T \beta_n)$  の漸近分布が、 $\hat{g}(x^T \beta_0)$  の漸近分布と一致するという話について：
- いま考えている  $\hat{g}(x^T \beta_n)$  というのは、generated regressor と呼ばれるものである。
- 一般に、generated regressor を用いて推定する場合、漸近分散を修正する必要がある。
- しかし、今のケースにおいては、 $\hat{g}(x^T \beta_n)$  の推定誤差がじゅうぶん大きいため、ラッキーなことに、そのような修正の必要がない。

## Generalized Cases?

- The direct average derivative estimation method discussed previously is applicable only when  $x$  is a  $q \times 1$  vector of continuous variables because the derivative w.r.t. discrete variables is not defined.
- Horowitz and Hardle (1996) discuss how direct (noniterative) estimation can be generalized to cases for which some components of  $x$  are discrete. Horowitz (2009) provides an excellent overview of this method.

- Nonparametric estimation in the 1st stage may suffer from the curse of dimensionality.
- In small-sample settings, the iterative method of Ichimura (1993) may be more appealing as it avoids having to conduct high-dimensional nonparametric estimation <sup>3</sup>.

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<sup>3</sup>あくまで, may be の話.

- They consider the problem of estimating a general partially linear single index model which contains both a partially linear model and a single index model as special cases.

# Bandwidth Selection

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## Bandwidth Selection for Ichimura's Estimator

- Recall that we assume in Assumption 8.4 that  $\frac{q \ln(h)}{nh^{3+\frac{3}{m-1}}} \rightarrow 0$  and  $nh^8 \rightarrow 0$  as  $n \rightarrow \infty$ , where  $m \geq 3$  is a positive integer whose specific value depends on the existence of the number of finite moments of  $Y$  along with the smoothness of the unknown function  $g(\cdot)$ .<sup>4</sup>
- The range of permissive smoothing parameters allows for optimal smoothing, i.e.,  $h = O(n^{-\frac{1}{5}})$ .<sup>5</sup>

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<sup>4</sup>Assumption 8.4 は,  $g$  をノンパラメトリックに推定することがパラメトリックパートの収束レートに影響を与えないための十分条件になっている.

<sup>5</sup>このオーダーで選んだ  $h$  は, Assumption 8.4 を満たしている.

- Our aim is to choose  $\hat{\beta}$  close to  $\beta_0$ , and  $h$  close to the value  $h_0$ , which minimize the average of

$$\mathbb{E}\{\hat{g}(X_i^T \beta_0 | X_i^T \beta_0) - g(X_i^T \beta_0)\}^2.$$

- Hardle, Hall and Ichimura (1993) suggest picking  $\beta$  and the bandwidth  $h$  jointly by minimization of  $S_n(\beta)$ .

# Bandwidth Selection for Ichimura's Estimator

- Recall the proof of **Theorem 8.1**. We have established the following decomposition of the least squares criterion:

$$\begin{aligned} S_n(\beta, h) &= \frac{1}{n} \sum_{i=1}^n (Y_i \hat{G}_{-i}(X_i^T \beta))^2 \\ &= \frac{1}{n} \sum_{i=1}^n (Y_i - G(X_i^T \beta))^2 \\ &\quad + \frac{1}{n} \sum_{i=1}^n (G_{-i}(X_i^T \beta_0) - g(X_i \beta_0))^2 + op(1) \\ &\equiv S(\beta) + T(h) + op(1). \end{aligned}$$

- Minimizing  $S_n(\beta, h)$  simultaneously over both  $(\beta, h) \in \mathcal{B}_n \times \mathcal{H}_n$  is equivalent to
  - first minimizing  $S(\beta)$  over  $\beta \in \mathcal{B}_n$ ; and
  - second minimizing  $T(h)$  over  $h \in \mathcal{H}_n$ .

- Let  $(\hat{\beta}, \hat{h})$  be the minimizers of  $S_n(\beta, h)$ .
- Suppose that we use the second order kernel. Hardle, Hall and Ichimura (1993) show that the MSE optimal bandwidth satisfies  $\hat{h} = O(n^{-\frac{1}{5}})$ , and  $\frac{\hat{h}}{h_0} \xrightarrow{p} 1$ .

## Bandwidth Selection for Ichimura's Estimator

- Compare the regularity conditions used in Ichimura (1993) with those in Hardle, Hall and Ichimura (1993).

### Ichimura (1993)

- A second order kernel is used.
- $h$  satisfies assumption 8.4.
- $\mathbb{E}[|Y^m|] < \infty$  for  $\exists m \geq 3$ .

### HHI (1993)

- A higher order kernel is needed <sup>a</sup>.
- $h = O(n^{-\frac{1}{5}})$ .
- $Y$  has moments of any order.

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<sup>a</sup>少なくとも 4 次.

- 左の仮定は識別のための仮定. 右の仮定は漸近分布を考えるために導入したより強い仮定.

# Bandwidth Selection for Average Derivative Estimator

- The estimation of  $\beta_0$  involves the  $q$ -dimensional multivariate nonparametric estimation of the first order derivatives.
- **Smoothing Parameters for  $\hat{f}_{-i}^{(1)}(X_i)$ :** Hardle and Tsybakov (1993) suggest to choose the smoothing parameters  $h_1, \dots, h_q$  to minimize MSE of  $\hat{\delta}$ .
- They show that the asymptotically optimal bandwidth is given by  $h_s = c_s n^{-\frac{2}{2q+v+2}}$ , for all  $s = 1, \dots, q$ , where  $c_s$  is the constant, and  $v$  is the order of kernel.
- Powell and Stoker (1996) provide a method for estimating  $c_s$ .
- Horowitz (2009) suggests to select  $h_s$  based on bootstrap resampling.

# Bandwidth Selection for Average Derivative Estimator

- **Smoothing Parameters for  $\hat{g}(X_i^T \beta_n)$ :** Once we select the optimal  $h_s$ 's, we can obtain an estimator of  $\beta$ . Let  $\beta_n$  denote a generic estimator.
- We estimate  $\mathbb{E}[y|x] = g(x^T \beta_0)$  by  $\hat{g}(x^T \beta_n, h) = \hat{g}(x^T \beta_n)$ . The smoothing parameter associated with the scalar index  $x^T \beta_n$  can be selected by least squares cross-validation:

$$\hat{h} = \arg \min_h \sum_{i=1}^n [Y_i - \hat{g}_{-i}(X_i^T \beta_n, h)]^2.$$

- Under some regularity conditions, the selection of  $h$  is of order  $Op(n^{-\frac{1}{5}})$ .

## Klein and Spady's (1993) Estimator

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# Semiparametric Binary Choice Model

- Consider the following binary choice model <sup>6</sup>:

$$Y_i = \begin{cases} 1 & \text{if } Y_i^* = \alpha + X_i^T \beta + \epsilon_i > 0, \\ 0 & \text{if } Y_i^* = \alpha + X_i^T \beta + \epsilon_i \leq 0. \end{cases}$$

- This model can be rewritten as

$$\begin{aligned} \mathbb{E}(Y_i \mid X_i) &= \mathbb{P}(Y_i = 1 \mid X_i) \\ &= \mathbb{P}(\alpha + X_i^T \beta + \epsilon_i > 0) \\ &= \mathbb{P}(\epsilon_i > -X_i^T \beta - \alpha) \equiv g(X_i^T \beta), \end{aligned}$$

which means that the binary choice model is a special case of the single index models.

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<sup>6</sup>(8.2) 式を参照せよと書いてあるが、 $\beta$  の識別のためには定数項の係数  $\alpha$  を抜いた方がよいのでは？

# Semiparametric Binary Choice Model

- Suppose that  $g(\cdot)$  were known. We would estimate  $\beta$  by maximum likelihood methods. The likelihood function would be

$$\begin{aligned} L^*(b) &= \mathbb{P}(\epsilon_i > -X_i^T b - \alpha)^{\sum_{i=1}^n Y_i} \\ &\quad \times \mathbb{P}(\epsilon_i \leq -X_i^T b - \alpha)^{\sum_{i=1}^n (1-Y_i)} \\ &= g(X_i^T b)^{\sum_{i=1}^n Y_i} \times \{1 - g(X_i^T b)\}^{\sum_{i=1}^n (1-Y_i)}, \end{aligned}$$

and then the log-likelihood function would be

$$L(b) = \sum_{i=1}^n [Y_i \log g(X_i^T b) + (1 - Y_i) \log(1 - g(X_i^T b))].$$

# Klein and Spady's (1993) Binary Choice Estimator

- In reality,  $g(\cdot)$  is unknown.
- Klein and Spady (1993) suggest to replace  $g(\cdot)$  with the leave-one-out NW estimator

$$\hat{g}_{-i}(X_i^T \beta) = \frac{\sum_{j \neq i} K\left(\frac{(X_i - X_j)^T \beta}{h}\right) Y_j}{\sum_{j \neq i} K\left(\frac{(X_i - X_j)^T \beta}{h}\right)}.$$

- Making this substitution, and adding a trimming function, this leads to the feasible likelihood criterion:

$$L(\beta) = \sum_{i=1}^n [Y_i \log \hat{g}_{-i}(X_i^T \beta) + (1 - Y_i) \log(1 - \hat{g}_{-i}(X_i^T \beta))] 1_i(b),$$

where the trimming indicator should not be a function of  $\beta$ , but instead of a preliminary estimator:

$$1_i(b) = 1 \left( \hat{f}_{X^T \tilde{\beta}}(X_i^T \tilde{\beta}) \geq b \right),$$

with a preliminary estimator  $\tilde{\beta}$  and density estimator  $\hat{f}_{X^T \tilde{\beta}}(\cdot)$ .

# Asymptotic Properties of Klein and Spady's Estimator

- The following asymptotic properties hold:
  - under some regularity conditions, and
  - assuming that the kernel  $k$  is of higher-order (must be of fourth-order).
- Define  $G(X_i^T \beta) = \mathbb{E}[g(X_i^T \beta_0) \mid X_i^T \beta]$ . Then we obtain the asymptotic distribution:

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} \text{Normal}(0, \Omega),$$

where the asymptotic variance is given by

$$\Omega = \mathbb{E} \left[ \frac{\partial}{\partial \beta} G(X_i^T \beta) \frac{\partial}{\partial \beta} G(X_i^T \beta)^T \frac{1}{g(X_i^T \beta_0)(1 - g(X_i^T \beta_0))} \right]^{-1}.$$

- Klein and Spady's (1993) estimator achieves **the semiparametric efficiency bound for the single-index binary choice model** (not for the general single-index model).

## Lewbel's (2000) Estimator

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- Consider the following binary choice model:

$$Y_i = 1(v_i + X_i^T \beta + \epsilon_i > 0),$$

where  $v_i$  is a (special) continuous regressor whose coefficient is normalized to be one and  $X_i$  is of dimension  $q$ .

- Let  $f(v|x)$  denote the conditional PDF of  $v_i$  given  $X_i$ .
- Let  $F_\epsilon(\epsilon | v, x)$  denote the conditional CDF of  $\epsilon_i$  conditioned on  $(v_i, X_i)$ .

## Lewbel's (2000) Estimator

- **Assumption:** Assume that  $F_\epsilon(\epsilon \mid v, x) = F_\epsilon(\epsilon \mid x)$ .
- In words, here we assume that, conditional on  $x$ ,  $\epsilon$  is independent of the special regressor  $v$ .
- We also introduce an orthogonality condition:  $\mathbb{E}(X_i \epsilon_i) = 0$ .
- **Identification:**

$$\beta = \mathbb{E}[X_i X_i^T]^{-1} \mathbb{E}[X_i \tilde{Y}_i], \text{ where } \tilde{Y}_i = \frac{Y_i - 1(v_i > 0)}{f(v_i \mid X_i)}.$$

- **Estimation:** Use the sample analogue of identification result, replacing the unknown quantity  $f(v_i \mid X_i)$  with its nonparametric kernel estimator <sup>7</sup>.

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<sup>7</sup>ということは, random denominator problem が生じる？

## Lewbel's (2000) Estimator: EXTENSION 1

- **Assumption:** There exists a  $p \times 1$  vector  $Z_i$  of IVs, which satisfies that  $\mathbb{E}(Z_i \epsilon_i) = 0$ ,  $\mathbb{E}(Z_i X_i^T)$  is non-singular, and  $F_{\epsilon x}(\epsilon, x \mid v, z) = F_{\epsilon x}(\epsilon, x \mid z)$ .
- We do not assume the orthogonality condition here.
- **Identification**<sup>8</sup>:

$$\beta = \mathbb{E}[Z_i X_i^T]^{-1} \mathbb{E}[Z_i \tilde{Y}_i], \text{ where } \tilde{Y}_i = \frac{Y_i - 1(v_i > 0)}{f(v_i \mid X_i)}.$$

- **Estimation:** Use the sample analogue of identification result, replacing the unknown quantity  $f(v_i \mid X_i)$  with its nonparametric kernel estimator<sup>9</sup>.

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<sup>8</sup>この形で識別しているということは、丁度識別のケースのみを考えている？

<sup>9</sup>ここでもやはり、random denominator problemが生じる？

## Lewbel's (2000) Estimator: EXTENSION 2

- Consider the ordered response model defined as

$$Y_i = \sum_{j=0}^{J-1} j 1(a_j < v_i + X_i^T \beta + \epsilon_i < a_{j+1}),$$

where  $a_0 = -\infty$  and  $a_J = +\infty$ .

- $Y_j$  is called the response variable, which takes values in the set  $\{0, 1, \dots, J-1\}$ .
- $Y_i = j$  if

$$a_j < v_i + X_i^T \beta + \epsilon_i < a_{j+1}.$$

- Let  $X_{1i} = 1$  be the intercept and  $\beta_1 = 0$ <sup>10</sup>.

<sup>10</sup>対応する係数がゼロでないと、インデックスモデルの識別条件に反する.

## Lewbel's (2000) Estimator: EXTENSION 2

- Set  $Y_{ji} = 1(Y_i \geq j)$  for  $j = 1, \dots, J - 1$ .
- Define  $\Delta = \mathbb{E}[X_i X_i^T]^{-1}$  and  $\Delta_j$  as the  $j$ th row of  $\Delta$ .
- **Identification:**

$$a_j = -\Delta_1 \mathbb{E} \left[ X_i \frac{Y_{ji} - 1(v_i > 0)}{f(v_i | X_i)} \right], \text{ for } j = 1, \dots, J - 1; \text{ and}$$

$$b_l = -\Delta_j \mathbb{E} \left[ X_i \frac{\frac{\sum_{j=1}^{J-1} Y_{ji}}{(J-1)} - 1(v_i > 0)}{f(v_i | X_i)} \right], \text{ for } l = 2, \dots, q.$$

- **Estimation:** Use the sample analogue of identification results, replacing  $f(v_i | X_i)$  with its nonparametric kernel estimator.
- **Further Extension:** These results can be extended to multinomial choices models, partially linear latent variable models, and threshold and censored regression models.

## Manski's (1975, 1985) Maximum Score Estimator

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# Manski's (1975, 1985) Maximum Score Estimator

- Consider the binary choice model:

$$Y_i = \begin{cases} 1 & \text{if } Y_i^* = X_i^T \beta + \epsilon_i > 0, \\ 0 & \text{if } Y_i^* = X_i^T \beta + \epsilon_i \leq 0, \end{cases}$$

$$\text{med}(\epsilon_i \mid X_i) = 0 \quad (\iff \text{med}(Y_i^* \mid X_i) = X_i^T \beta).$$

- Manski's maximum score estimator** is defined as

$$\hat{\beta}_M = \arg \max_{\beta} \sum_{i=1}^n Y_i 1(X_i^T \beta \geq 0) + (1 - Y_i) 1(X_i^T \beta < 0),$$

which is a LAD estimator of a linear median-regression model.

# Manski's (1975, 1985) Maximum Score Estimator

- Under some assumptions,  $\hat{\beta}_M$  has strong consistency.
- The rate of convergence is  $n^{-\frac{1}{3}}$  (Kim and Pollard, 1990).
- The limiting distribution is quite complex <sup>11</sup> and therefore not ideal for statistical inferences.
- To approximate the asymptotic distribution, Manski and Thompson (1986) use a bootstrap procedure.

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<sup>11</sup>A distribution of a maximum of a multidimensional Brownian motion with quadratic drift.

## Horowitz's (1992) Smoothed Maximum Score Estimator

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# Horowitz's (1992) Smoothed Maximum Score Estimator

- Consider the binary choice model:

$$Y_i = \begin{cases} 1 & \text{if } Y_i^* = X_i^T \beta + \epsilon_i > 0, \\ 0 & \text{if } Y_i^* = X_i^T \beta + \epsilon_i \leq 0, \end{cases}$$

$$\text{med}(\epsilon_i \mid X_i) = 0 \quad (\iff \text{med}(Y_i^* \mid X_i) = X_i^T \beta).$$

- Horowitz's modified maximum score estimator is defined as

$$\hat{\beta}_H = \arg \max_{\beta} \frac{1}{n} \sum_{i=1}^n (2Y_i - 1) G\left(\frac{X_i^T \beta}{h}\right),$$

where  $G(\cdot)$  is a  $p$ -times continuously differentiable CDF.

- Recall that Manski's criterion function has the indicator functions, which lead to the lack of continuity.
- Horowitz (1992) modifies Manski's criterion, replacing the indicator functions with a twice continuously differentiable function that retains the essential features.

## Horowitz's (1992) Smoothed Maximum Score Estimator

- Under some assumptions, as  $n \rightarrow \infty$ ;  $h = h_n > 0$ ; and  $h \rightarrow 0$ ,  
$$G\left(\frac{X_i^T \beta}{h}\right) \rightarrow 1(X_i^T \beta \geq 0).$$
- The convergence rate is  $\sqrt{nh}$ , and the asymptotic distribution is normal.
- Taking  $h = \left(\frac{c}{n}\right)^{\frac{1}{2p+1}}$  for some  $0 < c < \infty$ , the convergence rate becomes  $n^{-\frac{p}{2p+1}}$ .
- With sufficiently large  $p$ , the convergence rate becomes close to  $n^{-\frac{1}{2}}$ .

## **Han's (1987) Maximum Rank Estimator**

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# Han's (1987) Maximum Rank (Correlation) Estimator

- Consider the binary choice model:

$$Y_i = \begin{cases} 1 & \text{if } Y_i^* = X_i^T \beta + \epsilon_i > 0, \\ 0 & \text{if } Y_i^* = X_i^T \beta + \epsilon_i \leq 0, \end{cases}$$

$$\text{med}(\epsilon_i | X_i) = 0 \quad (\iff \text{med}(Y_i^* | X_i) = X_i^T \beta).$$

- Han's maximum rank correlation estimator<sup>12</sup> is defined as

$$\hat{\beta} = \arg \max_{\beta} \frac{2}{n(n-1)} \sum_{i=1}^n \sum_{j>i}^n 1(Y_i \geq Y_j) 1(X_i^T \beta \geq X_j^T \beta).$$

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<sup>12</sup>この推定量も indicator を用いて定義されているが、先ほどの maximum score estimator と違って、 $\sqrt{n}$  一致性をもつし、漸近分布は Normal である。  $\sum$  を 2 回とることで、ある種の smoothness が成立しているのだろうか？

# Han's (1987) Maximum Rank (Correlation) Estimator

- To motivate the maximum rank correlation estimator, observe that

$$\mathbb{P}(Y_i > Y_j \mid X_i, X_j) \geq \mathbb{P}(Y_i \leq Y_j \mid X_i, X_j),$$

whenever  $X_i^T \beta_0 \geq X_j^T \beta_0$ , which can be derived from the monotonicity of CDF and the independence of  $\epsilon_i$ 's and  $X_i$ 's.

- **Interpretation:** When  $X_i^T \beta_0 \geq X_j^T \beta_0$ , more likely than not  $Y_i \geq Y_j$ .
- Let  $G_H(\beta)$  denote the criterion function. Han (1987) shows that  $\mathbb{E}[G_H(\beta)]$  is maximized at  $\beta = \beta_0$ .
- Han (1987) also establishes the strong consistency.
- Sherman (1993) shows that the maximum rank correlation estimator is  $\sqrt{n}$ -consistent and has an asymptotically normal distribution<sup>13</sup>

<sup>13</sup> 目的関数のかたちを見てわかるように、これらの漸近的な性質は  $U$  統計量の漸近理論から derive されている。

## THE LIMITING DISTRIBUTION OF THE MAXIMUM RANK CORRELATION ESTIMATOR

By Robert P. Sherman<sup>1</sup>

### Abstract

Han's maximum rank correlation (MRC) estimator is shown to be  $\sqrt{n}$ -consistent and asymptotically normal. The proof rests on a general method for determining the asymptotic distribution of a maximization estimator, a simple U-statistic decomposition, and a uniform bound for degenerate U-processes. A consistent estimator of the asymptotic covariance matrix is provided, along with a result giving the explicit form of this matrix for any model within the scope of the MRC estimator. The latter result is applied to the binary choice model, and it is found that the MRC estimator does not achieve the semiparametric efficiency bound.

**KEYWORDS:** Generalized regression model, maximum rank correlation estimator, discontinuous criterion function, general method, U-statistic decomposition, uniform bound, empirical process, degenerate U-process, VC class, Euclidean class, numerical derivatives, semiparametric efficiency bound.

- Matzkin (1992) does not impose any parametric structure on either the systematic function of the observed exogenous variable or on the distribution of the random error term.

# Multinomial Discrete Choice Models

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# Multinomial Discrete Choice Models

- Consider the case where an individual faces  $J > 2$  choices.
- Define  $Y_{ij} = 1$  if individual  $i$  selects alternative  $j \in \{1, \dots, J\}$ ; and  $Y_{ij} = 0$  otherwise.
- Set  $F_{ij} = \mathbb{P}(Y_{ij} = 1 \mid X_i) = \mathbb{E}(Y_{ij} \mid X_i)$ .
- The multiple choice equation is given in

$$Y_{ij} = F_{ij} + \epsilon_{ij}.$$

- The likelihood function is

$$\sum_{i=1}^n \sum_{j=1}^J Y_{ij} \ln F_{ij}.$$

- **Semiparametric Approach:** Set

$$\begin{aligned} F_{ij} &= \mathbb{E}(Y_{ij} \mid X_{i1}^T \beta_1, \dots, X_{iJ}^T \beta_J) \\ &= g(X_{i1}^T \beta_1, \dots, X_{iJ}^T \beta_J), \end{aligned}$$

where the functional form of  $g(\cdot)$  is unknown.

- **Estimation:** Ichimura and Lee (1991), Lee (1995).

## **Ai's (1997) Semiparametric Maximum Likelihood Approach**

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*Econometrica*, Vol. 65, No. 4 (July, 1997), 933–963

## A SEMIPARAMETRIC MAXIMUM LIKELIHOOD ESTIMATOR

BY CHUNRONG AI<sup>1</sup>

This paper presents a procedure for analyzing a model in which the parameter vector has two parts: a finite-dimensional component  $\theta$  and a nonparametric component  $\lambda$ . The procedure does not require parametric modeling of  $\lambda$  but assumes that the true density of the data satisfies an index restriction. The idea is to construct a parametric model passing through the true model and to estimate  $\theta$  by setting the score for the parametric model to zero. The score is estimated nonparametrically and the estimator is shown to be  $\sqrt{N}$  consistent and asymptotically normal. The estimator is then shown to attain the semiparametric efficiency bound characterized in Begun et al. (1983) for multivariate nonlinear regression, simultaneous equations, partially specified regression, index regression, censored regression, switching regression, and disequilibrium models in which the error densities are unknown.

KEYWORDS: Kernel, semiparametric, nonparametric, asymptotic efficiency.

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